Solution Sheet 10

Exercise 10.1

Let (T_t) denote the transition semigroup induced by an SDE with coefficients satisfying the usual Lipschitz and linear growth conditions. Prove that (T_t) has the C_0 property, and that it is strongly continuous on C_0.

Proof. TBC

□

Exercise 10.2

Let (T_t) be a strongly continuous semigroup on C_0 with generator $\mathcal{L}$, induced by a cádlág Markov process (X_t). Suppose that $f, g \in C_0$ and that

$$M_t = f(X_t) - \int_0^t g(X_r)dr$$

is a martingale. Then $f \in D(\mathcal{L})$ and $\mathcal{L}f = g$.

Proof. By the martingale property, $\mathbb{E}(M_t) = \mathbb{E}(M_0) = f(X_0)$ and in particular

$$\mathbb{E}[f(X_t)] - \mathbb{E}\left[\int_0^t g(X_r)dr\right] = f(X_0).$$

Considering $X_0 = x$ and using that, by definition, $T_t f(x) = \mathbb{E}[f(X_t)]$, then

$$\frac{1}{t}\left(T_t f(x) - f(x)\right) = \frac{1}{t}\mathbb{E}\left[\int_0^t g(X_r)dr\right] = \frac{1}{t}\left[\int_0^t T_r g(x)dr\right].$$

This equality at all x implies the identity in C_0,

$$\frac{1}{t}\left(T_t f - f\right) = \frac{1}{t}\left[\int_0^t T_r g dr\right].$$

Since $r \mapsto T_r g$ is continuous on C_0, the right hand side converges, in C_0, to g which concludes the proof. □

Exercise 10.3

Let a, σ be given constants. Define the mapping $\mathcal{L}^*$ by

$$\mathcal{L}^* : \phi(x) \mapsto -(ax\phi(x))' + \frac{1}{2}(\sigma^2 x^2 \phi(x))''.$$

Propose a solution to the PDE

$$\frac{dp_t}{dt} = \mathcal{L}^* p_t.$$

Proof. We recall the Kolmogorov Forward Equation, that if X is a Markov Process whose transition semigroup has generator $\mathcal{L}$, and with law that is absolutely continuous with respect to the Lebesgue Measure, then its density p solves the PDE

$$\frac{dp_t}{dt} = \mathcal{L}^* p_t.$$

The idea is to look at the given PDE and try to find some SDE with generator $\mathcal{L}$ whose adjoint has the given form.

In this case, we would be able to conclude that the solution of the PDE is the density of the law of the solution to the SDE with generator $\mathcal{L}$. Recall that for an SDE

$$dX_t = f(X_T)dt + g(X_t)dW_t$$

then

$$\mathcal{L}^* : \phi \mapsto (-f\phi)' + \frac{1}{2}(g^2\phi)''$$

so in the given PDE,

$$f(x) = ax, \qquad g(x) = \sigma x.$$

Therefore the solution of the given PDE is the density of the law of X_t,

$$dX_t = aX_t + \sigma X_t dW_t$$

which has solution Geometric Brownian Motion, which is of log-normal distribution.

□

Exercise 10.4

Let (Y_t) be a continuous adapted stochastic process, such that for every $t \geq 0$ there exists a non-negative random variable $A_t \in L^1(\Omega; \mathbb{R})$ such that for any $\delta > 0$,

$$\frac{1}{\delta} \left| \mathbb{E}\left(Y_{t+\delta} - Y_t | \mathcal{F}_t\right) \right| \leq A_t$$

and

$$\lim_{\delta \to 0} \mathbb{E}\left(Y_{t+\delta} - Y_t | \mathcal{F}_t\right) = 0.$$

Prove that (Y_t) is a martingale.

Proof. Let $t \geq s$ and set $Z_t = \mathbb{E}(Y_t | \mathcal{F}_s)$. We wish to show that $Z_t = Z_s$, and do so by showing that it has null derivative from above at all t. Indeed,

$$\begin{aligned} \lim_{\delta \to 0} \frac{Z_{t+\delta} - Z_t}{\delta} &= \lim_{\delta \to 0} \frac{1}{\delta} \mathbb{E}\left(Y_{t+\delta} - Y_t | \mathcal{F}_s\right) \\ &= \lim_{\delta \to 0} \frac{1}{\delta} \mathbb{E}\left[\mathbb{E}\left(Y_{t+\delta} - Y_t | \mathcal{F}_t\right) | \mathcal{F}_s\right]. \end{aligned}$$

Now we use the Dominated Convergence Theorem, with domination from A_t, to take the limit inside which is zero from the second condition.

□

Exercise 10.5

For B a standard real valued Brownian Motion, show that $X_t := |B_t|$ is a Markov process, and write down its transition function. Prove that it induces a $C_0(\mathbb{R}_+)$ semi-group of generator $Łf = \frac{1}{2}f''$ with domain

$$\left\{f \in C^2(\mathbb{R}_+;\mathbb{R}) : f, f'' \in C_0\left(\mathbb{R}_+\right), f'(0) = 0\right\}.$$

Proof. Set T_t^B to be the semigroup of real Brownian motion. Given a bounded measurable function $f : \mathbb{R}_+ \mapsto \mathbb{R}$. By definition of Markov process,

$$\begin{aligned}
\mathbb{E}\left[f\left(X_{s+t}\right) \mid \mathcal{F}_s\right] = \mathbb{E}\left[f\left(|B_{s+t}|\right) \mid \mathcal{F}_s\right] = \mathbb{E}\left[f\left(|B_{s+t}|\right) \mid B_s\right]& \\
= \int_{-\infty}^{\infty} f(|y|)\frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{\left(y-B_s\right)^2}{2t}\right) dy& \\
= \int_0^{\infty} f(|y|)\frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{\left(y-B_s\right)^2}{2t}\right) dy + \int_{-\infty}^{0} f(|y|)\frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{\left(y-B_s\right)^2}{2t}\right) dy& \\
= \int_0^{\infty} f(|y|)\frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{\left(y-B_s\right)^2}{2t}\right) dy + \int_0^{\infty} f(|y|)\frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{\left(y+B_s\right)^2}{2t}\right) dy& \\
= \int_0^{\infty} f(y)\frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{\left(y-B_s\right)^2}{2t}\right) dy + \int_0^{\infty} f(y)\frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{\left(y+B_s\right)^2}{2t}\right) dy&
\end{aligned}$$

It's clear that

$$(t,x) \in \mathbb{R}_+ \times \mathbb{R}_+ \mapsto P_t(x,A) = \int_0^{\infty} \left(\frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(y-x)^2}{2t}\right) + \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(y+x)^2}{2t}\right)\right) 1_A(y)dy$$

is a measurable function. Thus, it suffices to show that $(T_t)_{t\geq 0}$ satisfy Chapman-Kolmogorov's identity. Let f be a bounded measuable function on $\mathbb{R}_+$. Define $g : \mathbb{R} \mapsto \mathbb{R}$ by $g(y) = f(|y|)$. By using similar argument as the proof of part 1, we have

$$T_t f(|x|) = T_t^B g(x) \quad \forall x \in \mathbb{R},$$

and therefore

$$\begin{aligned}
T_{t+s}f(x) = T_{t+s}^B g(x) = T_t^B T_s^B g(x) &= \int_{\mathbb{R}} T_s^B g(y)\frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(y-x)^2}{2t}\right) dy \\
&= \int_{\mathbb{R}_+} T_s^B g(y)\frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(y-x)^2}{2t}\right) dy + \int_{\mathbb{R}_-} T_s^B g(y)\frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(y-x)^2}{2t}\right) dy \\
&= \int_{\mathbb{R}_+} T_s^B g(y)\frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(y-x)^2}{2t}\right) dy + \int_{\mathbb{R}_+} T_s^B g(-y)\frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(y+x)^2}{2t}\right) dy \\
&= \int_{\mathbb{R}_+} T_s f(y)\frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(y-x)^2}{2t}\right) dy + \int_{\mathbb{R}_+} T_s f(y)\frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(y+x)^2}{2t}\right) dy \\
&= T_t T_s f(x) \quad \forall x \in \mathbb{R}_+.
\end{aligned}$$

Given $f \in C_0\left(\mathbb{R}_+\right)$. Then $g(x) \equiv f(|x|) \in C_0(\mathbb{R})$. Since T^B is a C_0 semigroup, we see that T is as well. Let f be a twice continuously differentiable function on $\mathbb{R}_+$, such that f and f'' belong to $C_0\left(\mathbb{R}_+\right)$ and $f'(0) = 0$. Define $g : \mathbb{R} \mapsto \mathbb{R}$ by $g(y) = f(|y|)$. Observe that

$$\lim_{x\to 0^+} \frac{g(x)-g(0)}{x} = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x} = f'(0).$$

and

$$\lim_{x\to 0^-} \frac{g(x)-g(0)}{x} = \lim_{x\to 0^-} \frac{f(-x)-f(0)}{x} = -f'(0).$$

Since $f'(0)=0, g'(0)$ exists and therefore

$$g'(y) = f'(|y|)\,\mathrm{sgn}(y)$$

and

$$g''(y) = f''(|y|),$$

where $\mathrm{sgn}(y) = 1_{\{y>0\}} - 1_{\{y<0\}}$. Thus g is a twice continuously differentiable function on $\mathbb{R}$, such that g and g'' belong to $C_0(\mathbb{R})$. Let L^B be the generator of $\left(T_t^B\right)_{t\geq 0}$. Then $L^B h = \frac{1}{2}h''$. We have

$$Lf(x) = \lim_{t\to 0} \frac{T_t f(x) - f(x)}{t} = \lim_{t\to 0} \frac{T_t^B g(x) - g(x)}{t} = \frac{1}{2}g''(x) = \frac{1}{2}f''(x) \quad \forall x \in \mathbb{R}_+$$

and therefore $Lf = \frac{1}{2}f''$. Conversely, assume that there exists $f \in C_0(\mathbb{R}_+) \cap D(L)$ such that $f'(0) \neq 0$. Then $g'(0)$ doesn't exist and $\lim_{t\to 0} \frac{T_t f(x) - f(x)}{t}$ exists for all $\forall x \in \mathbb{R}_+$. We see that

$$\lim_{t\to 0} \frac{T_t^B g(x) - g(x)}{t} = \lim_{t\to 0} \frac{T_t f(x) - f(x)}{t} = Lf(x) \quad \forall x \geq 0,$$
$$\lim_{t\to 0} \frac{T_t^B g(x) - g(x)}{t} = \lim_{t\to 0} \frac{T_t f(-x) - f(-x)}{t} = Lf(-x) \quad \forall x < 0,$$

and therefore $L^B g(x) = Lf(|x|)$ for all $x \in \mathbb{R}$. Since $Lf \in C_0(\mathbb{R}_+)$, we see that $L^B g \in C_0(\mathbb{R})$ and, hence, $g \in D\left(L^B\right) = \left\{h \in C^2(\mathbb{R}) \mid h \text{ and } h'' \in C_0(\mathbb{R})\right\}$ which is a contradiction. Thus, we reach the desired conclusion.

□